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NEAR-RESONANCE FREQUENCY CONTROL IN THE PRESENCE OF RANDOM PERTURBATIONS OF PARAMETERS†

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Weakly controlled resonance oscillations of a non-linear system are investigated. Small random perturbations results in a deviation of the frequencies from the resonance values. The aim of the control is to keep the frequencies in a small neighbourhood of the resonance surface. It is shown that small deviators of the frequencies from the given values can be found as a solution of a linearized diffusion equation. This enables the dynamic programming principle to be applied to the control problem. A bounded control which minimizes the mean time the system resides in the near-resonance domain is constructed. It is shown that the control is independent of the perturbations and the structure of the conservative part of the system. The frequency control for a system of coupled oscillators is considered as an example. © 2004 Elsevier Ltd. All rights reserved.

Standard transformations reduce the equations of the perturbed near-resonance motion of an oscillatory system to the equations of motion of an "equivalent pendulum", with domains of oscillations and rotations, and with the separatrix separating these domains [1, 2]. Passage across the separatrix from the domain of oscillations to the domain of rotations is associated with an extensive increase of the frequency deviation and breakdown of resonance. The aim of the control is to prevent the system from leaving an admissible domain under random perturbations.

This model enables the well-developed asymptotic methods for controlled oscillatory system to be employed [3, 4]. Control against escape from the resonance domain out the maximum time interval has been proposed [5]. Formally, the procedure developed could be applied to a wide range of systems. However, in practice, the investigate of a control problem for a stochastic resonance system over a large time interval is quite complicated. This paper considers more reverse control constraints, allowing the control problem to be studied over a relatively small time interval. The aim of the control is assumed to be to keep the system frequencies near resonance. The control problem for small deviations is reduced to the control problem for a linear system of variational equations for small deviations. The criterion and the constraint of the problem are also presented in terms of small deviations. A similar deterministic problem has been investigated in [3]. The solution of the variational equations for a stochastic system is approximated by a diffusion process [6, 7]. This enables the well-known problem of control for a linear stochastic system to be substituted for the original non-linear control problem. The solution of the stochastic control problem can be found by making use of the dynamic programming principle. The control constructed satisfies the specified constraints and maximizes the mean time the system resides in the near-resonance domain

1. THE MAIN EQUATIONS AND THE STATEMENT OF THE PROBLEM

A two-frequency system with a scalar slow variable will be considered in details.The extension to a multi-frequency system is discussed as an example in Section 3.

The equations of motion are reduced to the standard form with the slow and fast variables

$$\begin{aligned} \dot{y} &= \varepsilon f(y, \theta_1, \theta_2) + \varepsilon^n F(y, \theta_1, \theta_2)u + \varepsilon\Delta(y, \theta_1, \theta_2)\xi(t), \quad y \in Y, \quad u \in U \\ \dot{\theta}_i &= \omega_i(y) + \varepsilon f_i(y, \theta_1, \theta_2) + \varepsilon^n F_i(y, \theta_1, \theta_2)u + \varepsilon\Delta_i(y, \theta_1, \theta_2)\xi(t), \quad \theta_i (\text{mod} 2\pi) \\ i &= 1, 2 \end{aligned} \tag{1.1}$$

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where Y is an open domain, U is a closed domain in R_1, and $\varepsilon > 0$ is a small parameter. The exponent n in the coefficient ε^n is defined in such a way that the control would remain weak but counteracting the external perturbations in line with the requirements of the problem.

The right hand-sides of system (1.1) are assumed to be 2π-periodic in θ_1, θ_2 and smooth enough in all variables, for the solution of system (1.1) to exist and the required transformations to be valid for any admissible control. The perturbation $\xi(t)$ is considered as a zero-mean strong-mixing random process [7]. For instance, a Gaussian Markov process and a bounded random process with a rapidly decaying correlation function satisfy the requisite strong mixing condition [7].

Following the well-known approach [1], we will define the resonance relating between the system frequencies. We consider the time average $\Theta(y, \omega_1, \theta_2)$ for the function $f(y, \theta_1, \theta_2)$.

$$\Theta(y, \omega_1, \omega_2) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(y, \omega_1 t + \theta_1, \omega_2 t + \theta_2) dt$$

In the function $\Theta(y, \omega_1, \omega_2)$ is uniformly continuous in ω_1, ω_2 for all $y \in Y$, then the system is non-resonant, and $\Theta(y, \omega_1, \omega_2) = \langle f(y, \theta_1, \theta_2) \rangle$ where $\langle\cdot\rangle$ denotes the phase average. We will assume that the function Θ, considered as a function of the frequencies ω_1, ω_2, is discontinuous on the line

$$\rho(y) = m_1\omega_1(y) + m_2\omega_2(y) = 0 \tag{1.2}$$

for certain integers m_1, m_2 not equal to zero simultaneously. Equation (1.2) defines the resonance relation between the system frequencies. Let y^* be a unique isolated solution of Eq. (2.2) such that

$$\rho(y^*) = 0, \quad d\rho(y^*)dy = r \neq 0 \tag{1.3}$$

Also, we assume that the time average of the functions

$$F(y, \omega_1 t + \theta_1, \omega_2 t + \theta_2) \quad \text{and} \quad \Delta(y, \omega_1 t + \theta_1, \omega_2 t + \theta_2)\Delta(y, \theta_1, \theta_2)$$

do not generate new resonance relations in a small vicinity of the point y^*.

Let the unperturbed system exhibit a stable resonance mode of oscillations with frequencies $\omega_1(y^*)$ and $\omega_2(y^*)$, satisfying condition (1.2). The aim of the control is to keep the system frequencies in a neighbourhood of resonance in the presence of random perturbations, resulting in deviation of the variably y form the value y^* and leading to violation of the resonance condition (1.2).

We will formulate this requirement as an optimal control problem. We will define an admissible domain and find a control, which maximizes the mean time the system resides within the admissible domain.

Following the standard procedure [1, 2], we introduce the new variables υ and φ, corresponding to the frequency and phase deviation, respectively. We write

$$\begin{aligned} \mu\upsilon &= \rho(y) = m_1\omega_1(y) + m_2\omega_2(y), \quad \mu = \varepsilon^{1/2} \\ \varphi &= m_1\theta_1 + m_2\theta_2 \end{aligned} \tag{1.4}$$

It follows from definitions (1.3) and (1.4) that the relations

$$\begin{aligned} y &= Y(\mu\upsilon) = y^* + \mu y_1 + \mu^2 \ldots, y_1 = r^{-1}\upsilon \\ \theta_1 &= \theta, \quad \theta_2 = m_2^{-1}(\varphi - m_1\theta) \end{aligned} \tag{1.5}$$

hold in the near-resonance domain. Substituting Eqs (1.4) and (1.5) into system (1.1), we obtain the equations in the standard form with a small parameter μ

$$\begin{aligned} \dot{\upsilon} &= \mu[f^*(\varphi, \theta) + \Delta^*(\varphi, \theta)\xi(t)] + \mu^{2n-1}F^*(\varphi, \theta)u + \mu^2\Phi_1 \\ \dot{\varphi} &= \mu\upsilon + \mu^{2n}F_2^*(\varphi, \theta)u + \mu^2\Phi_2 \\ \dot{\theta} &= \omega^* + \mu\omega_1\upsilon + \mu^{2n}F_3^*(\varphi, \theta)u + \mu^2\Phi_3 \end{aligned} \tag{1.6}$$

where

$$\theta = \theta_1, \quad \omega^* = \omega(y^*), \quad \omega_1 = \omega_y(y^*)$$

The functions f^*, F^*, Δ^* are defined by the relations

$$f^*(\varphi, \theta) = r^{-1} f(y^*, \theta, \theta_2(\varphi, \theta)) \tag{1.7}$$

etc. The residual terms $\Phi_i(\upsilon, \varphi, \theta, u, \xi(t), \mu)$ on the right hand-sides of Eqs (1.7) vanish on taking the limit as $\mu \to 0$, and their explicit form is unimport for the further transformations.

We now define an admissible domain of motion. In system (1.6), we separate out the generic conservative subsystem

$$\upsilon' = \beta(\varphi), \quad \varphi' = \upsilon \tag{1.8}$$

where $\beta(\varphi) = \langle f(\varphi, \theta)\rangle$ and a prime denotes the derivative in the "slow time" $\tau = \mu t$. Equations (1.8) describe the motion of a pendulum with a periodic potential $U(\varphi)$ such that $U_\varphi(\varphi) = -\beta(\varphi)$. In the phase plane, the domain of the pendulum oscillations is associated with a closed domain Σ, bounded by the separatrix. This domain of motion is considered as admissible. Passage across the separatrix from the domain of oscillations to the domain of the pendulum rotation corresponds to an extensive increase in the frequency deviation, and is associated with the breakdown of resonance. Let the potential $U(\varphi)$ have a minimum at the point φ^*, that is, $\beta(\varphi^*) = 0$, $\beta_\varphi(\varphi^*) = c < 0$. The stable fixed point (y^*, φ^*) corresponds to resonance in the unperturbed system.

Hence, the control problem is reduced to control for the non-linear system (1.6) within the admissible domain $(\upsilon, \varphi) \in \Sigma$. A similar problem has been investigated and led to quite complicated results [5]. We shall simplify the control problem by considering small deviations from the unperturbed state. We define

$$\mu^{1/2} P = (\varphi - \varphi^*), \quad \mu^{1/2} Q = \upsilon \tag{1.9}$$

and write the control problems in terms of the variables P and Q. To keep the system frequencies in a small neighbourhood of resonance means to keep the process in a neighbourhood $D \subset \Sigma$ of the point $P = Q = 0$. The aim of the control is thus to maximize the mean time until the process $\{P(\tau, \mu), Q(\tau, \mu)\}$ reaches the boundary Γ of the domain D. The first exit time is defined as T^μ. The shape of the domain D depends on the constraints of the problem. We shall assume that D is an open simply-connected domain in R_2, and its closure $\bar{D}$ is symmetric about the origin, that is $\{P, Q\} \in \bar{D} \Leftrightarrow \{-P, -Q\} \in \bar{D}$. The control constraints take the form $|u| \leq U_0$. Under these assumptions, the criterion and the constraints of the problem can be written as

$$\begin{aligned} &J^\mu(u) = \mathrm{M} T^\mu \\ &T^\mu = \inf\{\tau: P(\tau, \mu), Q(\tau, \mu) \notin D / P(0, \mu), Q(0, \mu) \in D, |u| \leq U_0\} \end{aligned} \tag{1.10}$$

The optimal control u_{opt} is defined as

$$u_{\text{opt}} = \arg \max_{|u| \leq U_0} J^\mu(u) \tag{1.11}$$

We regard to the changes of variables (1.4) and (1.9), the control strategy can be interpreted as the locking of the system frequencies in a $\mu^{3/2}$-neighbourhood of the resonance point. This problem is meaningful, as the admissible domain Σ is of order μ, and the system remains within this domain, keeping away from the boundary.

Substituting relations (1.9) into system (1.6) and taking Eqs (1.8) into account, we write

$$\begin{aligned} Q' &= \mu^{-1/2}[\beta(\mu^{1/2}P + \varphi^*) - \beta(\varphi^*)] + \mu^{-1/2}\Delta^*(\mu^{1/2}P + \varphi^*, \theta)\xi(\tau/\mu) + \\ &+ \mu^{2n-5/2}F^*(\mu^{1/2}P + \varphi^*, \theta)u + \mu^{-1/2}b(\mu^{1/2}P + \varphi^*, \theta) + \mu^{1/2}\Phi_1, \quad Q(0) = 0 \\ P' &= Q + \mu^{2n-3/2}F_2^*(\mu^{1/2}P + \varphi^*, \theta)u + \mu^{1/2}\Phi_2, \quad P(0) = 0 \\ \theta' &= \mu^{-1}\omega^* + \mu^{2n-1}F_3^*(\mu^{1/2}P + \varphi^*, \theta)u + \mu^{1/2}\Phi_3 \end{aligned} \tag{1.12}$$

where

$$b(\varphi, \theta) = f^*(\varphi, \theta) - \beta(\varphi), \quad \langle b(\varphi, \theta) \rangle = 0$$

Hence, the control problem is reduced to the minimization of criterion (1.10) along the trajectories of system (1.12).

2. THE ASYMPTOTIC SOLUTION OF THE PROBLEM

The theorem on small (normal) deviations [6, 7] is used to analyse Eqs (1.12). We will formulate this theorem for an uncontrolled perturbed system and then extend it to controlled system (1.6).

We consider the equation

$$\dot{z} = \mu b(t, z) + B(t, z)\xi(t), \quad z(0) = z_0 \tag{2.1}$$

is the domain $z \in Z \in R_n$, $0 \leq t < \infty$. The vector $b(t, z)$ the matrix $B(t, z)$ are assumed to be uniformly continuous and twice continuously differentiable in z and uniformly continuous and bounded in the domain considered. The process $\xi(t)$ is a random strong mixing process in R_l, $M\xi(t) = 0$. Suppose the following conditions hold.

1. The limits

$$\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} b(t, z) dt = \beta(z), \quad \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \frac{\partial b(t, z)}{\partial z} dt = \frac{\partial \beta(z)}{\partial z}$$
$$\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} ds \int_{t_0}^{t_0+T} A_{ki}(s, t, z) dt = \alpha_{ki}(z) \tag{2.2}$$

where

$$A_{ki}(s, t, z) = \sum_{m, j = 1}^{l} B_{kj}(s, z) B_{mi}(t, z) M[\xi_j(s)\xi_m(s)]; \quad k, i = 1, \dots, n$$

exist uniformly for $z \in Z$, $t_0 \geq 0$.

2. The truncated system

$$\dot{z} = \mu\beta(z), \quad z(0) = z_0 \tag{2.3}$$

possesses the solution $z^*(\tau)$, $\tau = \mu t$.

Then the process

$$Z(\tau, \mu) = \frac{1}{\sqrt{\mu}}(z(t, \mu) - z^*(\tau)) \tag{2.4}$$

converges weakly as $\mu \to 0$ over the time interval $[0, T_0]$ to the Gaussian process $\zeta(\tau)$, satisfying the linear equation

$$\zeta'(\tau) = C(z^*(\tau))\zeta(\tau) + \sigma(z^*(\tau))w'(\tau), \quad \zeta(0) = 0 \tag{2.5}$$

where $w(\tau)$ is a standard Wiener process, the matrix of the drift coefficients is $C(z) = \partial\beta/\partial z$, and the diffusion matrix is defined as $\sigma(z)\sigma^T(z) = \alpha(z)$. The deterministic part of system (2.5) corresponds to the variational equation for truncated system (2.2). The second term on the right hand side of Eq. (2.5) is found taking into account the convergence of the perturbation $\mu^{-1/2}B(\tau/\mu, z)\xi(\tau/\mu)$ to the process $\sigma(z)w'(\tau)$ with the diffusion coefficient (2.2).

The conditions for the limit (2.2) to exist imply that the integrand in formulae (2.2) can be averaged. It has been shown [7] that the derivation of the limiting equation (2.5) involves two steps. First, the variational equation is derived for the non-averaged system and this equation is then averaged. Applications of the partial averaging procedure to stochastic systems [4, 8] implies that process (2.1) converges weakly to the solution $\zeta_\mu(\tau)$ of the partially averaged equation

$$\zeta_\mu'(\tau) = K(\tau/\mu, z^*(\tau))\zeta_\mu(\tau) + \sigma(z^*(\tau))w'(\tau), \quad \zeta_\mu(0) = 0 \tag{2.6}$$

where

$$K(t, z) = C(z) + k(t, z), \quad k(t, z) = \partial[b(t, z) - \beta(z)]/\partial z$$

In order to take the limit in Eqs (1.12) as $\mu \to 0$, we use the theorem on small deviation in the form (2.6). We put $n = 5/4$. Then, when $\mu \to 0$, the central term in the first equations of (1.12) is retained in the lending order approximation and is smaller in other equations. Comparing equalities (1.5), (2.1), (1.9) and (2.4) and making use of Eq. (2.6), we obtain that, for any admissible control, the process $\{P(\tau, \mu), Q(\tau, \mu)\}$ converges weakly as $\mu \to 0$, $0 \leqslant \tau < T_0$, to the solution $\{p(\tau, \mu), q(\tau, \mu)\}$ of the partially averaged system

$$\begin{aligned} p' &= q, \quad p(0) = 0 \\ q' &= cp + F_0(\theta)u + \sigma w'(\tau), \quad q(0) = 0 \end{aligned} \tag{2.7}$$

where

$$F_0(\theta) = F^*(\varphi^*, \theta), \quad \theta = \omega^*\tau/\mu$$

System (2.7) is similar to Eq. (2.6) with an additional term associated with the effect of the control. The diffusion coefficient is defined by the formula.

$$\sigma^2 = \int_{-\infty}^{\infty} R(s)\langle\Delta^*(\varphi^*, \theta)\Delta^*(\varphi^*, \theta + \omega^* s)\rangle ds \tag{2.8}$$

Under the assumptions of Section 1, the function $\Delta^*(\varphi^*, \theta)\Delta^*(\varphi^*, \theta + \omega^* s)$ does not yield additional resonance relations. This enables us to substitute the phase average for the time average and obtain formula (2.8).

Let τ^μ be the first moment the process $\{p(\tau, \mu), q(\tau, \mu)\}$ reaches the boundary Γ of the domain D, and $I^\mu(u) = M\tau^\mu$, $|u| \leqslant U_0$. We define the control

$$u^\mu = \arg\max_{|u| \le U_0} I^\mu(u) \tag{2.9}$$

It follows from the weak convergence $\{P, Q\} \to \{p, q\}$ that [7]

$$\left|I^\mu(u) - J^\mu(u)\right| \to 0, \quad \mu \to 0$$

for any admissible control $u \in U$. This implies [4]

$$\left|J^\mu(u^\mu) - J^\mu(u_{\text{opt}})\right| \to 0, \quad \mu \to 0 \tag{2.10}$$

that is the control u^μ is quasi-optimal with respect to the original problem. Hence, problem (1.10)–(1.12) can be replaced by a simpler problem (2.7), (2.9).

Problem (2.7), (2.9) is degenerate, as the perturbation is involved only in the second equation of system (2.7). However, the dynamic programming principle can be applied, with the solution interpreted in a generalized sense [9, 10].

The dynamic programming equation for problem (2.7), (2.9) takes the form

$$\begin{aligned} &\frac{\partial V}{\partial \tau} + cp\frac{\partial V}{\partial q} + q\frac{\partial V}{\partial p} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial q^2} + \max_{|u| \le U_0}\left[F_0(\theta)u\frac{\partial V}{\partial q}\right] = -1, \quad p, q \in D \\ &V(\tau, p, q) = 0, \quad p, q \in \Gamma \end{aligned} \tag{2.11}$$

where $\theta = \omega^*\tau/\mu$.

From Eq. (2.1), with due regard to the initial conditions (2.7), we deduce

$$u^\mu = U_0 \operatorname{sign}\left[F_0(\theta)\frac{\partial V}{\partial q}\right] \tag{2.12}$$

Substituting formula (2.12) into Eq. (2.11), we obtain the equation

$$\frac{\partial V}{\partial \tau}+cp\frac{\partial V}{\partial q}+q\frac{\partial V}{\partial p}+\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial q^2}+U_0\left|F_0(\theta)\frac{\partial V}{\partial q}\right| = -1, \quad p,q\in D$$
$$V(\tau,p,q) = 0, \quad p,q\in\Gamma \tag{2.13}$$

We shall demonstrate that the function V possesses a unique maximum at the point $p = q = 0$. In order to prove this, we replace Eq. (2.11) by a simpler homogeneous averaged equation. Averaging of Eq. (2.13) in the fast phase $\theta = \omega^*\tau/\mu$ leads to an equation which is independent of the slow variable τ.

$$cp\frac{\partial V^0}{\partial q}+q\frac{\partial V^0}{\partial p}+\frac{\sigma^2}{2}\frac{\partial^2 V^0}{\partial q^2}+U_0 f_0\left|\frac{\partial V^0}{\partial q}\right| = -1, \quad p,q\in D$$
$$V^0(p,q) = 0, \quad p,q\in\Gamma \tag{2.14}$$

where $f_0 = \langle|F_0(\theta)|\rangle$. The uniform convergence [11]

$$V(\tau,p,q)\underset{\mu\to 0}{\to}V^0(p,q), \quad p,q\in\bar{D}, \quad 0\le\tau\le\tau^\mu \tag{2.15}$$

leads to the relation

$$I^\mu(u^\mu) = V(0,0,0)\underset{\mu\to 0}{\to}V^0(0,0) \tag{2.16}$$

It follows from Eqs (2.11)–(2.14) and the symmetry of the domain $\bar{D}$ that $V^0(-p,-q) = V^0(p,q)$. If a bounded solution of Eq. (2.14) exists, the series expansion of the even function $V^0(p,q)$ only involves even positive exponents of the arguments. This implies

$$\frac{\partial V^0}{\partial q} = \frac{\partial V^0}{\partial p} = 0, \quad p = q = 0 \tag{2.17}$$

In turn, we deduce from relations (2.4) and (2.17) that $\partial^2V^0/\partial q^2 = -2/\sigma^2 < 0$ for $p = q = 0$. This means that the function $V^0(p,q)$ has a maximum in q at the point $p = q = 0$.

We will show that the point $p = q = 0$ is the unique maximum of the function in the domain D. Let another point in D exist, at which the function $V^0(p,q)$ is a maximum. Then an intermediate point exists, at which the function $V^0(p,q)$ is a minimum, that is $\partial V^0/\partial q = \partial V^0/\partial p = 0$. However, $\partial^2V^0/\partial q^2 > 0$ for $p = \bar{p}, q = \bar{q}$. The last inequality contradicts Eq. (2.14). Thus $p = q = 0$ is the unique maximum of the function $V^0(p,q)$ in the domain D.

In particular, the maximum at $q = 0$ implies that

$$\operatorname{sign}\frac{\partial V^0}{\partial q} = -\operatorname{sign} q \tag{2.18}$$

for all $p,q\in D$. Using relation (2.18), we reduce Eq. (2.14) to the form

$$cp\frac{\partial V^0}{\partial q}+q\frac{\partial V^0}{\partial p}+\frac{\sigma^2}{2}\frac{\partial^2 V^0}{\partial q^2}-U_0 f_0\frac{\partial V^0}{\partial q}\operatorname{sign} q = -1, \quad p,q\in D$$
$$V^0(p,q) = 0, \quad p,q\in\Gamma \tag{2.19}$$

Relations (2.12), (2.18) and (2.19) are used to construct feedback control. We will consider two versions of the feedback control following from these relations

$$1)\ u_1(\upsilon,\theta) = -U_0\operatorname{sign}F_0(\theta)\operatorname{sign}\upsilon$$
$$2)\ u_2(\upsilon,y,\theta_1,\theta_2) = -U_0\operatorname{sign}[r^{-1}F(y,\theta_1,\theta_2)]\operatorname{sign}\upsilon \tag{2.20}$$

where, by equalities (1.4) and (1.7)

$$F_0(\theta) = r^{-1}F(y^*,\theta_1,\theta_2(\theta_1,\varphi^*))$$

We will show that the controls u_1 and u_2 are quasi-optimal with respect to the perturbed system (1.12). To prove this, we calculate the mean time MT_1^μ and MT_2^μ required to reach the boundary of the domain D by the trajectories of system (1.2) in case $u = u_1$ or $u = u_2$, respectively. Substituting u_1 or u_2 into system (1.1) and reproducing the transformations of Sections 1 and 2, we obtain the weak convergence as $\mu \to 0$ of the solution $P(\tau, \mu)$, $Q(\tau, \mu)$ of system (1.12) to the solution $p_0(\tau)$, $q_0(\tau)$ of the averaged system

$$\begin{aligned} p_0' &= q_0 \quad p_0(0) = 0 \\ q_0' &= cp_0 - f_0U_0 \operatorname{sign} q_0 + \sigma w'(\tau), \quad q_0(0) = 0 \end{aligned} \tag{2.21}$$

System (2.21) is similar to system (2.7) but it is obtained by averaging in all right-hand side terms, including the control terms. In particular, the weak convergence $\{P, Q\} \to \{p, q\}$, $\{P, Q\} \to \{p_0, q_0\}$ implies [7]

$$MT_1^\mu \to MT_0, \quad MT_2^\mu \to MT_0, \quad \mu \to 0 \tag{2.22}$$

where T_0 is the first moment the process $\{p_0(\tau), q_0(\tau)\}$ reaches the boundary of the domain D, that is

$$T_0 = \inf\{\tau \colon p_0(\tau), q_0(\tau) \notin D\}$$

In turn, it follows from Eqs (2.19) and (2.21) that

$$MT_0 = V^0(0, 0) \tag{2.23}$$

Comparing relations (2.10), (2.16), (2.22) and (2.23), we obtain that

$$\left|J^\mu(u_{1,2}) - J^\mu(u_{\text{opt}})\right| \to 0, \quad \mu \to 0 \tag{2.24}$$

The implies that the controls (2.20) are quasi-optimal and independent of the perturbations and the structure of the uncontrolled system. The physical interpretation of the solution follows from Eqs (2.21): in consideration of the "equivalent pendulum", the controls u_1 or u_2 can be interpreted as Coulomb friction with the largest admissible coefficient.

If a different criterion and other control constraints are chosen, control may be dependent on the system structure and perturbations. However, in this case, too, the control problem for the original non-linear system can be reduced to a similar control problem for linear system (2.5).

3. FREQUENCY CONTROL FOR THE FORCE MOTION OF A NON-LINEAR OSCILLATOR

As an example, we shall consider the control problem for resonance oscillations of a one-degree-of-freedom non-linear system subject to random fluctuations of the natural frequency. The equation of the controlled motion takes the form

$$\ddot{x} + \varepsilon n\dot{x} + \phi(x) + \varepsilon\xi(t)g(x) = \varepsilon a \sin\Omega t + \varepsilon^{5/4}u \tag{3.1}$$

Here $\phi(x) = d\Pi(x)/dx$, $\Pi(x)$ is the potential of the conservative part of the system and $\xi(t)$ is a random perturbation satisfying the assumptions of Section 1. The control u is constructed as in Section 1.

We put $\dot{x} = z$ and introduce the new variables y and θ_2 by the formulae [1]

$$\begin{aligned} y &= \frac{1}{2}z^2 + \Pi(x), \quad \frac{\partial\theta_2}{\partial x} = \frac{\omega(y)}{z(y, x)} \\ z(y, x) &= \pm\sqrt{2(y - \Pi(x))}, \quad \omega(y) = \frac{2\pi}{T(y)} \end{aligned} \tag{3.2}$$

The period of motion is defined as

$$T(y) = \oint\limits_{y = \text{const}} \frac{dx}{z(y, x)}$$

The second relation in (3.2) yields the formal dependence $x = X(y, \theta_2)$ and, respectively, $z(y, x) = Z(y, \theta_2)$.

Substituting of the variables (3.2) transforms Eq. (3.1) into the system, similar to (1.1),

$$\begin{aligned}\dot{y} &= \varepsilon Y(y, \theta_1, \theta_2, \xi(t), u)Z(y, \theta_2)\\ \dot{\theta}_2 &= \omega(y) + \varepsilon\frac{\partial\omega}{\partial y}Y(y, \theta_1, \theta_2, \xi(t), u)Z(y, \theta_2)\\ \dot{\theta}_1 &= \Omega\end{aligned} \tag{3.3}$$

where

$$Y(y, \theta_1, \theta_2, \xi(t), u) = -nZ(y, \theta_2) - \xi(t)g(X(y, \theta_2)) + a\sin\theta_1 + \varepsilon^{1/4}u \tag{3.4}$$

If follows from Eqs (3.3) and (3.4) that an infinite number of lines of discontinuity similar to (1.2) can exist if the function $Z(y, \theta_2)$ is a 2π-periodic in θ_2. We will study the first harmonic resonance, for which conditions (1.2) and (1.3) take the form

$$\begin{aligned}\rho(y) &= \omega(y) - \Omega = 0, \quad \rho(y^*) = 0\\ d\rho(y^*)/dy &= d\omega(y^*)/dy = r \neq 0\end{aligned} \tag{3.5}$$

We use transformation (1.4) in order to analyse small deviations from resonance. This yields

$$\mu v = \rho(y) = \omega(y) - \Omega, \quad \varphi = \theta_1 - \theta_2, \quad \theta_1 = \theta \tag{3.6}$$

As in Section 1, we seek the control strategy which counteracts frequency deviation from resonance. Small deviations are defined by formulae (1.9) and the criterion and the constraints of the problem are defined by formulae (1.10). Substituting the new variables (3.6) into Eqs (3.3), using relations (1.9), and reproducing the transformations of Sections 1 and 2, we find that control (2.12) or (2.20) are determined by the function

$$F(y, \theta_1, \theta_2) = Z(y, \theta) = \dot{x} \tag{3.7}$$

From relations (3.7) and (3.20) we drive the control synthesis in the form

$$u = u_2 = -U_0\,\mathrm{sign}(r^{-1}\dot{x})\,\mathrm{sign}[\omega(y) - \Omega] \tag{3.8}$$

where $r = \omega_y(y^*)$. The sign of the coefficient r can often be found without direct calculation of the frequency $\omega(y)$: $r > 0$, if the system is "hard", and $r < 0$, if the system is "soft" in a small neighbourhood of the point y^*.

We will consider a different scheme of excitation for resonance oscillations. Suppose periodic excitation cannot be applied directly to the non-linear oscillator. A weak signal is amplified by a resonance circuit and transmitted to the input of the controlled non-linear system by interconnecting circuits. The equations of motion of two loosely coupled oscillators are written in the form

$$\begin{aligned}\ddot{x} + \varepsilon n\dot{x} + \phi(x) + \varepsilon\xi(t)g(x) &= \varepsilon^{5/4}u + \varepsilon f(\psi, \dot{\psi})\\ \ddot{\psi} + \varepsilon b\dot{\psi} + \Omega^2\psi &= \varepsilon a\sin\Omega t + \varepsilon s(x, \dot{x})\end{aligned} \tag{3.9}$$

The functions $f(\psi, \dot{\psi})$ and $s(x, \dot{x})$ represent the interconnections of both subsystems.

We reduce system (3.9) to the standard form. By using formulae (3.2), we replace the variables $x, \dot{x}$ by the variables y, θ_2. Then, we define the variables ψ, $\dot{\psi}$ by the standard transformation

$$\psi = R\cos\theta_1, \quad \dot{\psi} = -\Omega R\sin\theta_1 \tag{3.10}$$

Substituting expressions (3.2) and (3.10) into Eqs (3.9) using the notation of Section 1, we reduce system (3.9) to the form

$$\begin{aligned}\dot{R} &= -\frac{\varepsilon}{\Omega}[\Psi(R, \theta_1, \theta_3) + S(y, \theta_2)]\sin\theta_1\\ \dot{y} &= \varepsilon[f(y, \theta_2) + \Theta(R, \theta_1)Z(y, \theta_2) + \Delta(y, \theta_2)\xi(t) + \varepsilon^{1/4}F(y, \theta_2)u]\end{aligned}$$

$$\begin{aligned}\dot{\theta}_1 &= \Omega - \frac{\varepsilon}{\Omega R}[\Psi(R, \theta_1, \theta_3) + S(y, \theta_2)]\cos\theta_1 \\ \dot{\theta}_2 &= \omega(y) + \varepsilon\frac{\partial\omega}{\partial y}[f(y, \theta_2) + \Theta(R, \theta_1)Z(y, \theta_2) + \Delta(y, \theta_2)\xi(t) + \varepsilon^{1/4}F(y, \theta_2)u] \\ \dot{\theta}_3 &= \Omega\end{aligned} \tag{3.11}$$

where $Z(y, \theta_2) = \dot{x}$, and the other coefficients take the form

$$\begin{aligned}\Psi(R, \theta_1, \theta_3) &= b\Omega R\sin\theta_1 + a\sin\theta_3 \\ f(y, \theta_2) &= -nZ^2(y, \theta_2), \quad \Delta(y, \theta_2) = -g(X(y, \theta_2))Z(y, \theta_2) \\ F(y, \theta_2) &= Z(y, \theta_2) \\ S(y, \theta_2) &= s(X(y, \theta_2), Z(y, \theta_2)), \quad \Theta(R, \theta_1) = f(R\cos\theta_1 - \Omega R\sin\theta_1)\end{aligned} \tag{3.12}$$

The non-linear connections can generate an infinite number of resonance relations, similar to (1.2). Suppose the aim of the control is to sustain the first-harmonic resonance oscillations. In this case, the resonance condition (3.5) remains valid.

We introduce the variables

$$\begin{aligned}\varphi &= \theta_2 - \theta_1, \quad \varphi_1 = \theta_1 - \theta_3, \quad \theta_2 = \theta \\ \mu\upsilon &= \rho(y) = \omega(y) - \Omega, \quad \mu = \varepsilon^{1/2}\end{aligned} \tag{3.13}$$

Substitution of relations (3.5) and (3.13) in system (3.13) yields the equations

$$\begin{aligned}\dot{R} &= -\mu^2[\Psi_1(R, \theta + \varphi, \varphi_1) + S_1(\theta, \varphi)] + \mu^3\ldots \\ \dot{\varphi}_1 &= -\mu^2[\Psi_2(R, \theta + \varphi, \varphi_1) + S_2(\theta, \varphi)] + \mu^3\ldots \\ \dot{\upsilon} &= \mu[f^*(\theta) + Q^*(R, \theta, \varphi)] + \mu\Delta^*(\theta)\xi(t) + \mu^{3/2}F^*(\theta)u\ldots \\ \dot{\varphi} &= \mu\upsilon + \mu^2\ldots \\ \dot{\theta} &= \Omega + \mu\ldots\end{aligned} \tag{3.14}$$

The coefficients f^*, Δ^*. F^* are determined as in Section 1, the function $Q^* = r^{-1}\Theta(R, \theta - \varphi)Z(y^*, \theta)$, and the functions Ψ_i and S_i are obtained by substituting the formulae $y = y^*$, $\theta_1 = \theta + \varphi$, $\theta_3 = \theta + \varphi + \varphi_1$ into the right-hand sides of the corresponding equations in system (3.11).

System (3.14) involves the slow variables R and φ_1, the "semi-fast" variables v and φ, and the variable θ. The motion can be analysed by the technique of successive averaging, with due regard to the second-approximation terms. However, the fixed points can be found from the first-approximation system.

By analogy with Section 1, we separate out a truncated averaged system from system (3.14). Averaging the right hand-sides of system (3.14) in the fast phase θ, taking into account the functions (3.12) and neglecting the small terms of higher orders, we obtain

$$\begin{aligned}\dot{R} &= -\mu^2[\psi_1(R, \varphi_1) + s_1(\varphi)] \\ \dot{\varphi}_1 &= -\mu^2[\psi_2(R, \varphi_1) + s_2(\varphi)] \\ \dot{\upsilon} &= \mu[\beta_0 + \beta_1(R, \varphi)] \\ \dot{\varphi} &= \mu\upsilon\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\beta_0 &= \langle f^*(\theta)\rangle, \quad \beta_1(R, \varphi) = \langle Q^*(R, \theta, \varphi)\rangle \\ \psi_i(R, \varphi_1) &= \langle\Psi_i(R, \theta + \varphi, \varphi_1)\rangle, \quad s_i(\varphi) = \langle S_i(\theta, \varphi)\rangle\end{aligned}$$

The fixed points can be found as the solution of the system

$$\upsilon = 0, \quad \beta_0 + \beta_1(R, \varphi) = 0, \quad \psi_i(R, \varphi_1) + s_i(\varphi) = 0 \tag{3.16}$$

Separation of the terms which depend on the phases φ and φ_1 eliminates the secondary resonances from consideration. The stability of the steady states is investigated by the well-known method [1].

We will find the control which counteracts the frequency deviations from resonance. We will investigate the deviations from the stationary point R^*, φ^*, φ_1^* as $t \sim 1/\mu$, and $\mu \to 0$. Since the rates of change of the slow and "semi-fast" variables are different, the rates of the deviations should be different. Comparing the exponents of the small parameter in Eqs (3.14), we represent the small deviations in the form

$$\mu^{1/2}Q = \upsilon, \quad \mu^{1/2}P = \varphi - \varphi^*, \quad \mu^{3/2}H = R - R^*, \quad \mu^{3/2}G = \varphi_1 - \varphi_1^* \tag{3.17}$$

This implies that the deviations of R^* and φ_1^* from the fixed values are small compared with the deviations of the "semi-fast" variables.

We substitute the variables (3.17) into system (3.14), repeat the transformations of Section 1 and 2, and take into account the exponents of the small parameter. As a result, we obtain that as $\mu \to 0$ the equations for the variables P and Q are separated and become independent of the variables G and H. This implies that the criterion and the constraints of the problem can be defined by formulae (1.10), independent of the variables G and H.

It follows from the Theorem of Section 2 that, as $\mu \to 0$, $0 \le \tau \le T_0$, the process $\{P, Q, G, H\}$ converges to the solution $[p, q, g, h]$ of an approximated linearized system. As in Eqs (2.7), the deterministic part of the linearized system corresponds to the variational equations for the averaged system (3.15), and the control and excitation forces are taken into account. With due regard to relations (3.17) we obtain that, as $\mu \to 0$, the approximating equations take the form

$$\begin{aligned} p' &= q, \quad p(0) = 0 \\ q' &= cp + F_0(\theta)u + \sigma w'(\tau), \quad q(0) = 0 \\ g' &= k_1 p, \quad h' = k_2 p \end{aligned} \tag{3.18}$$

where

$$c = \partial\beta_1(R^*, \varphi)/\partial\varphi, \quad F_0(\theta) = Z(y^*, \theta), \quad k_i = ds_i(\varphi^*)/d\varphi$$

The prime denotes a derivative $d/d\tau$, $\tau = \mu t$.

The equations for the variables p and q are separated, that is, the previous conclusions, relating to the construction of the quasi-optimal control in terms of the variables P and Q, remain valid. In particular, this yields the feedback control in the form

$$u = u_2 = -U_0 \operatorname{sign}(r^{-1}\dot{x}) \operatorname{sign}[\omega(y) - \Omega] \tag{3.19}$$

This means that, under the assumptions of Section 1, the quasi-optimal control (3.19) is independent of the perturbation and the parameters of the linear subsystem. The only required parameter is the sign of the coefficient r. This result remains valid for a deterministic system.

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